3729. Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.

If a,b,c are the side lengths of a triangle, prove that

$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ca}+\frac{a+b}{c^2+ab}\le\frac{3(a+b+c)}{ab+bc+ca}.$$

Solution by Arkady Alt , San Jose ,California, USA.

Let $S=a+b+c, P:=ab+bc+ca, Q:=abc$ then

$(a^2+bc)(b^2+ca)(c^2+ab)=2a^2b^2c^2+abc(a^3+b^3+c^3)+a^3b^3+a^3c^3+b^3c^3=$

$2Q^2+Q(3Q+S^3-3SP)+3Q^2+P^3-3SPQ=P^3-6PQS+8Q^2+QS^3$

and $\sum_{cyc}(a+b)(a^2+bc)(b^2+ca)=\sum_{cyc}ab(a^3+b^3)+\sum_{cyc}a^2b^2(a+b)+$

$2abc\sum_{cyc}a^2+2abc\sum_{cyc}ab=(ab+bc+ca)(a^3+b^3+c^3)+$

$abc(a^2+b^2+c^2)+(a^2b^2+b^2c^2+c^2a^2)(a+b+c)+abc(ab+bc+ca)=$

$P(S^3+3Q-3SP)+Q(S^2-2P)+S(P^2-2SQ)+QP=PS^3+2QP-QS^2-2P^2S$

and original inequality becomes

(**1**) $$\frac{PS^3+2QP-QS^2-2P^2S}{P^3-6PQS+8Q^2+QS^3}\le\frac{3S}{P}.$$

Let $s:=\frac{a+b+c}{2}$ (semiperimeter) and $x:=s-a, y:=s-b, z:=s-c$. Assuming, due to homogeneity of original inequality, that $s=1$ we obtain $a=1-x, b=1-y, c=1-z$ where $x,y,z>0$ and $x+y+z=1$.

Denoting $p:=xy+yz+zx, q=xyz$ we obtain $S=2, P=1+p, Q=p-q$,

$P^3-6PQS+8Q^2+QS^3=(1+p)^3-12(p-q)(1+p)+8(p-q)^2+8(p-q)=$

$p^3-p^2-4pq-p+8q^2+4q+1,\ PS^3+2QP-QS^2-2P^2S=$

$8(1+p)+2(p-q)(1+p)-4(p-q)-4(1+p)^2=2q-2p-2pq-2p^2+4$ and (**1**)$\Leftrightarrow$

$$\frac{2q-2p-2pq-2p^2+4}{p^3-p^2-4pq-p+8q^2+4q+1}\le\frac{6}{1+p}\Leftrightarrow\frac{q-p-pq-p^2+2}{p^3-p^2-4pq-p+8q^2+4q+1}\le\frac{3}{1+p}\Leftrightarrow$$

$3(p^3-p^2-4pq-p+8q^2+4q+1)-(1+p)(q-p-pq-p^2+2)\ge 0\Leftrightarrow$

$4p^3-p^2-4p+1+24q^2+p^2q-12pq+11q\ge 0\Leftrightarrow$

(**2**) $(p-1)(4p-1)(p+1)+24q^2+q(p^2-12p+11)\ge 0,$

where $p=xy+yz+zx=\frac{(x+y+z)^2}{3}=\frac{1}{3}$ and $q\ge\frac{(1-p)(4p-1)}{6}$ (is Schur Inequality $\sum_{cyc}x^2(x-y)(x-z)\ge 0$ in p,q-notation and normalized by $x+y+z=1$).

Obvious that $p^2-12p+11>0$ for $0<p\le\frac{1}{3}$. Hence $24q^2+q(p^2-12p+11)\uparrow(q\ge 0)$.

Since $q\ge\max\left\{0,\frac{(1-p)(4p-1)}{6}\right\}$ then for $0<p\le\frac{1}{4}$ we have $q\ge 0$ and, therefore,

$(p-1)(4p-1)(p+1)+24q^2+q(p^2-12p+11)\ge(p-1)(4p-1)(p+1)\ge 0;$

If $\frac{1}{4}<p\le\frac{1}{3}$ then $q\ge q_*:=\frac{(1-p)(4p-1)}{6}$ and

$(p-1)(4p-1)(p+1)+24q^2+q(p^2-12p+11)\ge(p-1)(4p-1)(p+1)+$

$24q_*^2+q_*(p^2-12p+11)=(p-1)(4p-1)(p+1)+$

$$\frac{2(1-p)^2(4p-1)^2}{3}+\frac{(1-p)(4p-1)(p-1)(p-11)}{6}=$$

$$\frac{(1-p)(4p-1)}{6}(-6(p+1)+4(1-p)(4p-1)+(p-1)(p-11)) =$$
$$\frac{(1-p)(4p-1)(5p+1)(1-3p)}{6} \geq 0.$$

Equality occurs iff $p = \frac{1}{3}, q = \frac{1}{27}$ or $p = \frac{1}{4}, q = 0.$

In original notations in the first case we get $a = b = c$ and in the second case – degenerated triangle $a = 0, b = c$ and two more cyclic.